arXiv:hep-th/9412060v3 16 Dec 1994

TWI-94-29

QUANTIZATION OF THE LIE ALGEBRA SO(2N+1) AND OF THE LIE SUPERALGEBRA OSP(1/2N) WITH PREOSCILLATOR GENERATORS*

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Abstract

The Lie algebra $so(2n+1)$ and the Lie superalgebra $osp(1/2n)$ are quantized in terms of $3n$ generators, called preoscillator generators. Apart from n "Cartan" elements the preoscillator generators are deformed para-Fermi operators in the case of $so(2n+1)$ and deformed para-Bose operators in the case of $osp(1/2n)$. The corresponding deformed universal enveloping algebras $U_q[so(2n+1)]$ and $U_q[osp(1/2n)]$ are the same as those defined in terms of Chevalley operators. The name "preoscillator" is to indicate that in a certain representation these operators reduce to the known deformed Fermi and Bose operators.

1. Preoscillator realization and oscillator representations of osp(2n+1/2m) and of some of its subalgbras [1]

The Lie superalgebra $osp(2n+1/2m) \equiv B(n/m)$ can be defined as the set of all matrices of the form (T=transposition)[2]:

$$\begin{pmatrix} a & b & u & x & x_1 \\ c & -a^T & v & y & y_1 \\ -v^T & -u^T & 0 & z & z_1 \\ y_1^T & x_1^T & z_1^T & d & e \\ -y^T & -x^T & -z^T & f & -d^T \end{pmatrix}, \tag{1}$$

* An invited talk at the Meeting of the FNRS Contact Group on Mathematical Physics, Université Libre de Bruxelles, October 1994

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where a is any $(n \times n)$ matrix; b and c are skew symmetric $(n \times n)$ matrices; d is any $(m \times m)$; e and f are symmetric $(m \times m)$ matrices; u and v are $(n \times 1)$ columns; x, x_1, y, y_1 are $(n \times m)$ matrices and z, z_1 are $(1 \times m)$ rows.

The even subalgebra consists of all matrices with

$$x = x_1 = y = y_1 = z = z_1 = 0,$$

namely

$$\begin{pmatrix} a & b & u & 0 & 0 \\ c & -a^T & v & 0 & 0 \\ -v^t & -u^T & 0 & 0 & 0 \\ 0 & 0 & 0 & d & e \\ 0 & 0 & 0 & f & -d^T \end{pmatrix} \tag{2}$$

and it is isomorphic to the Lie algebra $so(2n+1) \oplus sp(2m)$. The odd subspace consists of all matrices

$$\begin{pmatrix} 0 & 0 & 0 & x & x_1 \\ 0 & 0 & 0 & y & y_1 \\ 0 & 0 & 0 & z & z_1 \\ y_1^T & x_1^T & z_1^T & 0 & 0 \\ -y^T & -x^T & -z^T & 0 & 0 \end{pmatrix}. \tag{3}$$

The product (= the supercommutator) is defined on any two homogeneous elements a, b from $osp(2n+1/2m)$ as

$$[\![a, b]\!] = ab - (-1)^{deg(a)deg(b)} ba. \tag{4}$$

An important role in the construction plays the $2(n+m)-$dimensional $\mathbf{Z}_2-$graded subspace $G(n/m)$ consisting of all matrices

$$\begin{pmatrix} 0 & 0 & u & 0 & 0 \\ 0 & 0 & v & 0 & 0 \\ -v^T & -u^T & 0 & z & z_1 \\ 0 & 0 & z_1^T & 0 & 0 \\ 0 & 0 & -z^T & 0 & 0 \end{pmatrix}. \tag{5}$$

Label the rows and the columns with the indices

$$A, B = -2n, -2n+1, \ldots, -2, -1, 0, 1, 2, \ldots, 2m$$

and let $e_{A,B}$ be a matrix with 1 at the intersaction of the $A^{th}-$row and the $B^{th}-$column and zero elsewhere. Choose as a basis in $G(n/m)$ the following elements (matrices):

$$\begin{aligned} C_i^-(1) &\equiv B_i^- = \sqrt{2}(e_{0,i} - e_{i+m,0}),\ i = 1, \ldots, m, \\ C_i^+(1) &\equiv B_i^+ = \sqrt{2}(e_{0,i+m} + e_{i,0}), \\ C_j^-(0) &\equiv F_j^- = \sqrt{2}(e_{-j,0} - e_{0,-j-n}),\ j = 1, \ldots, n, \\ C_j^+(0) &\equiv F_j^+ = \sqrt{2}(e_{0,-j} - e_{-j-n,0}). \end{aligned} \tag{6}$$

The elements $C_i^{\pm}(1)$ are odd and $C_j^{\pm}(0)$ - even. We call all these elements creation and annihilation operators (CAOs) of $osp(2n+1/2m)$.

Proposition 1 [1]: The LS $osp(2n+1/2m)$ is generated from its creation and annihilation operators.

It turns out that already the supercommutators of all CAOs give the lacking basis elements. Therefore

$$lin.env.\{[\![C_i^{\xi}(\alpha), C_j^{\eta}(\beta)]\!],\ C_k^{\varepsilon}(\gamma)|\forall i,j,k;\ \xi,\eta,\varepsilon=\pm;\ \alpha,\beta,\gamma\in \mathbb{Z}_2\} = osp(2n+1/2m). \quad (7)$$

Hence any further supercommutator between

$$[\![C_i^{\xi}(\alpha), C_j^{\eta}(\beta)]\!] \quad \text{and} \quad C_k^{\varepsilon}(\gamma)$$

is a linear combination of the same type of elements. The more precise computation gives:

$$[\![[\![C_i^{\xi}(\alpha), C_j^{\eta}(\beta)]\!], C_k^{\varepsilon}(\gamma)]\!] = 2\varepsilon^{\gamma}\delta_{\beta\gamma}\delta_{jk}\delta_{\varepsilon,-\eta}C_i^{\xi}(\alpha) - 2\varepsilon^{\gamma}(-1)^{\beta\gamma}\delta_{\alpha\gamma}\delta_{ik}\delta_{\varepsilon,-\xi}C_j^{\eta}(\beta), \quad (8)$$

where $\xi,\eta,\varepsilon=\pm,\ \alpha,\beta,\gamma\in\mathbb{Z}_2$ and i,j,k take all possible values.

In the case $\alpha=\beta=\gamma=0$ (8) reduces to

$$[[F_i^{\xi}, F_j^{\eta}], F_k^{\varepsilon}] = 2\delta_{jk}\delta_{\varepsilon,-\eta}F_i^{\xi} - 2\delta_{ik}\delta_{\varepsilon,-\xi}F_j^{\eta}, \quad (9)$$

whereas for $\alpha=\beta=\gamma=1$ it gives

$$[\{B_i^{\xi}, B_j^{\eta}\}, B_k^{\varepsilon}] = 2\varepsilon\delta_{jk}\delta_{\varepsilon,-\eta}B_i^{\xi} + 2\varepsilon\delta_{ik}\delta_{\varepsilon,-\xi}B_j^{\eta}. \quad (10)$$

Equations (9) and (10) are the defining relations for the para-Fermi and para-Bose operators, respectively [3].

One can also say that the relations (8) define a structure of a Lie-super triple system on $G(n/m)$ [4] with a triple product

$$G(n/m)\otimes G(n/m)\otimes G(n/m)\rightarrow G(n/m)$$

defined from (8).

It is important to point out that the relations (8) define completely the supercommutation relations between all generators. In order to show this one has to use simply the (graded) Jacoby indentity:

$$[\![[\![a,b]\!],[\![c,d]\!]]\!] = [\![[\![[\![a,b]\!],c]\!],d]\!] + (-1)^{(deg(a)+deg(b))deg(c))}[\![c,[\![[\![a,b]\!],d]\!]]\!].$$

Hence the triple relations (8) define completely the Lie superalgebra $osp(2n+1/2m)$.

We have given a definition of $osp(2n+1/2m)$ in terms of a particular $(2n+2m+1)\times(2n+2m+1)$ matrix realization. The triple relations (8) are however representation independent.

In order to give a representation independent definition, denote by U the associative superalgebra with unity, (abstract) generators

$$C_j^{\pm}(0) \equiv F_j^{\pm}, \quad j=1,2,\ldots,n \tag{11}$$

as even elements and

$$C_i^{\pm}(1) \equiv B_i^{\pm}, \quad i=1,2,\ldots,m \tag{12}$$

as odd elements, which obey the triple relations (8). The supercommutator $[\![a,b]\!]$ between any homogeneous elements a,b is defined through the associative multiplication in U:

$$[\![a,b]\!] = ab - (-1)^{deg(a)deg(b)}ba. \tag{13}$$

With respect to the binary operation $[\![\ ,\ ]\!]$ U is a Lie superalgebra.

Proposition 2. The (finite-dimensional) subspace

$$B(n/m) = lin.env.\{[\![C_i^{\xi}(\alpha), C_j^{\eta}(\beta)]\!],\ C_k^{\varepsilon}(\gamma) | \forall i,j,k;\ \xi,\eta,\varepsilon=\pm;\ \alpha,\beta,\gamma\in \mathbb{Z}_2\} \tag{14}$$

of U is a subalgebra of the Lie superalgebra U, isomorphic to $osp(2n+1/2m)$. U is the universal enveloping algebra $U[osp(2n+1/2m)]$ of $osp(2n+1/2m)$.

Observe that the para-Bose (pB) operators neither commute nor anticommute with the para-Fermi (pF) operators.

One can express any other generator (or, more generally, any other element from U) in terms of the preoscillator generators (= pB & pF operators).

In particular, one can express the Chevalley generators of $osp(2n+1/2m)$. To this end set:

$$\begin{aligned} C_i^{\pm}(0) &\equiv C_i^{\pm}, \quad i=1,2,\ldots,n, \\ C_j^{\pm}(1) &\equiv C_{j+n}^{\pm}, \quad j=1,2,\ldots,m. \end{aligned} \tag{15}$$

Then one has

$$\begin{aligned} e_i &= \frac{1}{2}[\![C_i^-, C_{i+1}^+]\!],\ i=1,\ldots,n+m-1, \\ e_{n+m} &= -\frac{1}{\sqrt{2}}C_{n+m}^-, \\ f_i &= \frac{1}{2}[\![C_{i+1}^-, C_i^+]\!],\ i=1,\ldots,n+m-1, \\ f_{n+m} &= \frac{1}{\sqrt{2}}C_{n+m}^+, \\ h_i &= \frac{1}{2}[\![C_{i+1}^+, C_{i+1}^-]\!] - [\![C_i^+, C_i^-]\!],\ i=1,\ldots,n+m-1, \\ h_{n+m} &= -\frac{1}{2}[\![C_{n+m}^+, C_{n+m}^-]\!]. \end{aligned} \tag{16}$$

As is well known the Chevalley generators describe completely the corresponding algebra, in this case $osp(2n+1/2m)$. The preoscillator generators give an alternative description. In terms of the latter it is easy to express various subalgebras of $osp(2n+1/2m)$:

$$osp(2n+1/2m) = lin.env.\{C_i^\xi,\ [\![C_j^\eta, C_k^\epsilon]\!]|i,j,k=1,\ldots,n+m,\ \xi,\eta,\epsilon=\pm\}; \tag{17}$$

$$osp(2n/2m) = lin.env.\{[\![C_j^\eta, C_k^\epsilon]\!]|j,k=1,\ldots,n+m,\ \eta,\epsilon=\pm\}; \tag{18}$$

$$gl(n/m) = lin.env.\{[\![C_j^+, C_k^-]\!]|j,k=1,\ldots,n+m\}; \tag{19}$$

$$\begin{aligned} so(2n+1) &= lin.env.\{C_i^\xi,\ [C_j^\eta, C_k^\epsilon]|i,j,k=1,\ldots,n,\ \xi,\eta,\epsilon=\pm\} \\ &= lin.env.\{F_i^\xi,\ [F_j^\eta, F_k^\epsilon]|i,j,k=1,\ldots,n,\ \xi,\eta,\epsilon=\pm\}; \end{aligned} \tag{20}$$

$$\begin{aligned} sp(2m) &= lin.env.\{\{C_j^\eta, C_k^\epsilon\}|j,k=n+1,\ldots,n+m,\ \eta,\epsilon=\pm\} \\ &= lin.env.\{\{B_j^\eta, B_k^\epsilon\}|j,k=1,\ldots,m,\ \eta,\epsilon=\pm\}; \end{aligned} \tag{21}$$

$$\begin{aligned} gl(n) &= lin.env.\{[C_j^+, C_k^-]|j,k=1,\ldots,n\} \\ &= lin.env.\{[F_j^+, F_k^-]|j,k=1,\ldots,n\}; \end{aligned} \tag{22}$$

$$\begin{aligned} gl(m) &= lin.env.\{\{C_j^+, C_k^-\}|j,k=n+1,\ldots,n+m\} \\ &= lin.env.\{\{B_j^+, B_k^-\}|j,k=1,\ldots,m\}; \end{aligned} \tag{23}$$

Proposition 3 [1]. Let

$$f_i^\pm,\ i=1,\ldots,n \quad \text{be Fermi operators and}$$

$$b_j^\pm,\ j=1,\ldots,m \quad \text{be Bose operators}$$

under the additional requirement that the Bose operators **anticommute** with the Fermi operators,

$$\{f_i^\xi, b_j^\eta\}=0 \quad \forall i,j \text{ and } \xi,\eta.$$

Then the map

$$F_i^\xi \ \rightarrow\ f_i^\xi, \quad B_j^\eta \ \rightarrow\ b_j^\eta$$

defines a representation of $osp(2n+1/2m)$.

This construction is rather unconvenional in the following sense:

1. The Bose operators are odd, fermionic variables and the Fermi operators are even, bosonic variables.

2. The Bose and the Fermi operators anticommute.

To the same conclusion arrived recently also Okubo [4] and Macfarlane [5].

In view of the above construction it is natural to ask questions like:

- Can one define deformed preoscillator operators and describe the quantum algebra $U_q[osp(2n+1/2m)]$ in terms of these operators?
- If yes, then what is the relation of these operators to the deformed Bose and Fermi operators, introduced in [6-9]?
- How do the deformed preoscillator and the deformed oscillator realizations look like?

The general answer to all of the above questions in unknown. At present some of these questions can be answered for the deformed $so(2n+1)$ and $osp(1/2m)$ (super)algebras.

We proceed now to outline how this can be done first for the odd-orthogonal Lie algebra $so(2n+1)$ for any n.

2. Quantization of so(2n+1) with deformed para-Fermi operators [10]

So far the algebra $so(2n+1)$ and more precisely - its universal enveloping algebra $U[so(2n+1)]$ - has been quantized in terms of its Chevalley generators. Let $\hat{e}_i, \hat{f}_i, h_i,\ i = 1,\ldots,n$ be the nondeformed Chevalley generators. Then $U[so(2n+1)]$ is the associative unital (=with unity) algebra with (free) generators

$$\hat{e}_i,\ \hat{f}_i,\ h_i,\quad i = 1,\ldots,n,$$

which obey the Cartan relations

$$[h_i, h_j] = 0,\quad [h_i, \hat{e}_i] = \alpha_{ij}\hat{e}_i,\quad [h_i, \hat{f}_i] = -\alpha_{ij}\hat{f}_i,\quad [\hat{e}_i, \hat{f}] = \delta_{ij}h_i \tag{24}$$

and the Serre relations

$$\begin{gathered}
[\hat{e}_i, \hat{e}_j] = 0,\quad [\hat{f}_i, \hat{f}_j] = 0,\quad |i-j| > 1,\\
[\hat{e}_i, [\hat{e}_i, \hat{e}_{i\pm1}]] = 0\ \ [\hat{f}_i, [\hat{f}_i, \hat{f}_{i\pm1}]] = 0,\ \ i \neq n,\\
[\hat{e}_n, [\hat{e}_n, [\hat{e}_n, \hat{e}_{n-1}]]] = 0,\ \ [\hat{f}_n, [\hat{f}_n, [\hat{f}_n, \hat{f}_{n-1}]]] = 0.
\end{gathered} \tag{25}$$

The Cartan matrix (α_{ij}) is taken to be symmetric with

$$\begin{gathered}
\alpha_{nn} = 1,\ \alpha_{ii} = 2,\ i = 1,\ldots,n-1,\\
\alpha_{j,j+1} = \alpha_{j+1,j} = -1,\ j = 1,\ldots,n-1,
\end{gathered} \tag{26}$$

and all other $\alpha_{ij} = 0$.

On the other hand, as we have alreay indicated, $U[so(2n+1)]$ is the algebra of n-pairs of pF operators, ($\xi, \eta, \epsilon = \pm$ or ± 1, $i, j, k = 1, 2, \ldots, n$):

$$[[\hat{F}_i^{\xi}, \hat{F}_j^{\eta}], \hat{F}_k^{\epsilon}] = \frac{1}{2}(\epsilon - \eta)^2 \delta_{jk} \hat{F}_i^{\xi} - \frac{1}{2}(\epsilon - \xi)^2 \delta_{ik} \hat{F}_j^{\eta}. \tag{27}$$

The expressions of the Chevalley generators in terms of the pF operators read:

$$\begin{aligned}
&\hat{e}_n = \tfrac{1}{\sqrt{2}} \hat{F}_n^{-}, && \hat{e}_i = \tfrac{1}{2}[\hat{F}_i^{-}, \hat{F}_{i+1}^{+}], \\
&\hat{f}_n = \tfrac{1}{\sqrt{2}} \hat{F}_n^{+}, && \hat{f}_i = \tfrac{1}{2}[\hat{F}_{i+1}^{-}, \hat{F}_i^{+}], \\
&h_n = \tfrac{1}{2}[\hat{F}_n^{-}, \hat{F}_n^{+}], && h_i = \tfrac{1}{2}[\hat{F}_i^{-}, \hat{F}_i^{+}] - \tfrac{1}{2}[\hat{F}_{i+1}^{-}, \hat{F}_{i+1}^{+}].
\end{aligned} \tag{28}$$

The inverse relations are not that simple ($i = 1, \ldots, n-1$):

$$\begin{aligned}
&\hat{F}_i^{-} = \sqrt{2}[\hat{e}_i, [\hat{e}_{i+1}, [\hat{e}_{i+2}, [\ldots, [\hat{e}_{n-2}, [\hat{e}_{n-1}, \hat{e}_n]] \ldots], \\
&\hat{F}_i^{+} = \sqrt{2}[\ldots[\hat{f}_n, \hat{f}_{n-1}], \hat{f}_{n-2}], \ldots], \hat{f}_{i+2}], \hat{f}_{i+1}], \hat{f}_i], \\
&\hat{F}_n^{+} = \sqrt{2}\hat{f}_n, \quad \hat{F}_n^{-} = \sqrt{2}\hat{e}_n.
\end{aligned} \tag{29}$$

Following Khoroshkin and Tolstoy [11] we define the deformed UEA

$$U_q[so(2n+1)] \equiv U_q$$

in terms of its Chevalley generators as follows: U_q is the (free unital) associative algebra with Chevalley generators ($i = 1, \ldots, n$)

$$e_i,\ f_i,\ k_i = q^{h_i},\ \bar{k}_i \equiv k_i^{-1} = q^{-h_i}, \tag{30}$$

which satisfy the Cartan relations

$$\begin{gathered}
k_i k_i^{-1} = k_i^{-1} k_i = 1, \ \ k_i k_j = k_j k_i, \\
k_i e_j = q^{\alpha_{ij}} e_j k_i, \ \ k_i f_j = q^{-\alpha_{ij}} f_j k_i, \\
[e_i, f_j] = \delta_{ij} \frac{k_i - \bar{k}_i}{q - \bar{q}}
\end{gathered} \tag{31}$$

and the Serre relations ($\bar{q} \equiv q^{-1}$)

$$\begin{gathered}
[e_i, e_j] = 0, \quad [f_i, f_j] = 0, \quad |i - j| > 1, \\
[e_i, [e_i, e_{i\pm 1}]_{\bar{q}}]_q = 0, \quad [f_i, [f_i, f_{i\pm 1}]_{\bar{q}}]_q = 0, \quad i \neq n, \\
[e_n, [e_n, [e_n, e_{n-1}]_{\bar{q}}]]_q = 0, \\
[f_n, [f_n, [f_n, f_{n-1}]_{\bar{q}}]]_q = 0.
\end{gathered} \tag{32}$$

Here and throughout

$$[a,b]_{q^n}=ab-q^n ba \tag{33}$$

and it is assumed that the deformation parameter q is any complex number except

$$q=0,\ \ q=1,\ \ q^2=1.$$

The deformed pF operators $F_i^{\pm}$ are defined as follows:

$$\begin{aligned}
&F_i^- =\sqrt{2}[e_i,[e_{i+1},[e_{i+2},[\ldots,[e_{n-1},e_n]_{\bar q}]_{\bar q}\ldots]_{\bar q},\\
&F_n^- =\sqrt{2}e_n,\\
&F_i^+ =\sqrt{2}[\ldots[f_n,f_{n-1}]_q,f_{n-2}]_q,\ldots]_q,f_{i+1}]_q,f_i]_q,\\
&F_n^+ =\sqrt{2}f_n.
\end{aligned} \tag{34}$$

Let

$$L_i=k_ik_{i+1}\ldots k_n,\ \ i=1,\ldots,n.$$

We call the operators

$$F_i^{\pm},\ \ L_i\ \ i=1,\ldots,n \tag{35}$$

preoscillator generators of $U_q[so(2n+1)]$.

Proposition 4. The defining relations (31), (32) of $U_q[so(2n+1)]$ in terms of its Chevalley generators (30) hold if and only if the preoscillator generators (35) satisfy the relations:

$$\begin{aligned}
&L_iL_i^{-1}=L_i^{-1}L_i=1,\ \ L_iL_j=L_jL_i,\\
&L_iF_j^{\pm}=q^{\mp\delta_{ij}}F_j^{\pm}L_i,\quad i,j=1,\ldots,n,\\
&[F_i^-,F_i^+]=2\tfrac{L_i-\bar L_i}{q-\bar q},\quad i=1,\ldots,n,\\
&[[F_i^{-\eta},F_{i\pm1}^{\eta}],F_j^{-\eta}]_{q^{\pm\delta_{ij}}}=2\delta_{j,i\pm1}L_j^{\pm\eta}F_i^{-\eta},\quad \eta=\pm,\\
&[F_n^{\xi},[F_n^{\xi},F_{n-1}^{\xi}]]_{\bar q}=0,\quad \xi=\pm.
\end{aligned} \tag{36}$$

Therefore $U_q[so(2n+1)]$ can be viewed as a free associative unital algebra of the preoscillator generators with relations (36).

In terms of the preoscillator generators it is very easy to write an analogue of the Cartan-Weyl basis:

Proposition 5. The operators ($\xi=\pm$)

$$L_i,\ F_i^{\pm},\ [F_i^-,F_j^+],\ [F_p^{\xi},F_q^{\xi}],\ i\neq j,\ i,j,p,q=1,\ldots,n$$

are an analogue of the Cartan-Weyl generators for $so(2n+1)$. In terms of these generators one can introduce a basis in $U_q[so(2n+1)]$.

3. Quantization of osp(1/2n) with deformed para-Bose operators

A. First realization [12-15]

We proceed first to introduce $U_q \equiv U_q[osp(1/2n)]$ in terms of its Chevalley generators. The Cartan matrix (α_{ij}) is chosen as before, i.e., as a $n \times n$ symmetric matrix with

$$\alpha_{nn} = 1,\ \alpha_{ii} = 2,\ i = 1, \ldots, n-1,$$

$$\alpha_{j,j+1} = \alpha_{j+1,j} = -1,\ j = 1, \ldots, n-1,$$

and all other $\alpha_{ij} = 0$.

Then U_q is the associative superalgebra with Chevalley generators

$$e_i,\ f_i,\ k_i = q^{h_i},\ i = 1, \ldots, n,$$

which satisfy the Cartan-Kac relations

$$\begin{gathered}
k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \quad i,j = 1, \ldots, n, \\
k_i e_j = q^{\alpha_{ij}} e_j k_i, \quad k_i f_j = q^{-\alpha_{ij}} f_j k_i, \quad i,j = 1, \ldots, n, \\
\{e_n, f_n\} = \frac{k_n - k_n^{-1}}{q - q^{-1}}, \\
[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} \quad \forall\, i,j \text{ except } i = j = n,
\end{gathered} \tag{37}$$

the Serre relations for the simple positive root vectors

$$\begin{gathered}
[e_i, e_j] = 0, \quad \text{if} \quad i,j = 1, \ldots, n \quad \text{and} \quad |i-j| > 1, \\
e_i^2 e_{i+1} - (q + q^{-1}) e_i e_{i+1} e_i + e_{i+1} e_i^2 = 0, \quad i = 1, \ldots, n-1, \\
e_i^2 e_{i-1} - (q + q^{-1}) e_i e_{i-1} e_i + e_{i-1} e_i^2 = 0, \quad i = 2, \ldots, n-1, \\
e_n^3 e_{n-1} + (1 - q - q^{-1})(e_n^2 e_{n-1} e_n + e_n e_{n-1} e_n^2) + e_{n-1} e_n^3 = 0,
\end{gathered} \tag{38}$$

and the Serre relations obtained from above by replacing everywhere e_i by f_i.

The grading on U_q is induced from the requirement that the generators $e_n,\ f_n$ are odd and all other generators are even.

Passing to the para-Bose operators we first observe that the nondeformed operators are defined with the relations ($\xi, \eta, \epsilon = \pm$ or ± 1, $i,j,k = 1, 2, \ldots, n$)

$$[\{\hat{A}_i^{\xi}, \hat{A}_j^{\eta}\}, \hat{A}_k^{\epsilon}] = (\epsilon - \xi)\delta_{ik}\hat{A}_j^{\eta} + (\epsilon - \eta)\delta_{jk}\hat{A}_i^{\xi}. \tag{39}$$

For the deformed pB operators we set:

$$\begin{aligned}
A_i^- &= -\sqrt{2}[e_i, [e_{i+1}, [\ldots, [e_{n-2}, [e_{n-1}, e_n]_{q^{-1}}]_{q^{-1}} \ldots]_{q^{-1}}, \quad i = 1, \ldots, n-1), \\
A_n^- &= -\sqrt{2}e_n, \\
A_i^+ &= \sqrt{2}[\ldots [f_n, f_{n-1}]_q, f_{n-2}]_q, \ldots]_q, f_{i+1}]_q, f_i]_q, \quad i = 1, \ldots, n-1), \\
A_n^+ &= \sqrt{2}f_n.
\end{aligned} \tag{40}$$

Introduce also n even "Cartan" elements

$$\begin{aligned}
&L_i = k_i k_{i+1} \ldots k_n = q^{H_i}, \\
&H_i = h_i + \ldots + h_n, \quad i = 1, \ldots, n.
\end{aligned} \tag{41}$$

The expressions of the Chevalley generators in terms of the preoscillator generators read ($i \neq n$):

$$\begin{aligned}
&e_n = -(2)^{-1/2}A_n^-, \quad e_i = -\frac{q}{2}\{A_i^-, A_{i+1}^+\}L_{i+1}^{-1}, \\
&f_n = (2)^{-1/2}A_n^+, \quad f_i = -\frac{1}{2q}L_{i+1}\{A_i^+, A_{i+1}^-\}.
\end{aligned} \tag{42}$$

As in the pF case we call the operators

$$L_i^{\pm 1}, \ A_i^{\pm}, \quad i = 1, \ldots, n \tag{43}$$

preoscillator generators.

Proposition 5. The relations of $U_q[osp(1/2n)]$ in terms of its Chevalley generators hold if and only if the preoscillator generators satisfy the relations:

$$\begin{aligned}
&L_i L_i^{-1} = L_i^{-1} L_i = 1, \ \ L_i L_j = L_j L_i, \\
&L_i A_j^{\pm} = q^{\mp\delta_{ij}} A_j^{\pm} L_i, \quad i, j = 1, \ldots, n, \\
&\{A_i^-, A_i^+\} = -2\tfrac{L_i - \bar{L}_i}{q - \bar{q}}, \quad i = 1, \ldots, n, \\
&[\{A_i^{-\eta}, A_{i\pm 1}^{\eta}\}, A_j^{-\eta}]_{q^{\pm\delta_{ij}}} = -2\eta\delta_{j,i\pm 1} L_j^{\pm\eta} A_i^{-\eta}, \quad \eta = \pm, \\
&[\{A_{n-1}^{\xi}, A_n^{\xi}\}, A_n^{\xi}]]_q = 0, \quad \xi = \pm.
\end{aligned} \tag{44}$$

The relations (44) replace completely the Cartan-Kac relations (37) and the Serre relations (38). They give an alternative definition of $U_q[osp(1/2n)]$. In terms of the preoscillator generators it is easy to define all Cartan-Weyl generators:

$$L_i^{\pm 1},\ A_i^{\pm},\ \{A_i^-, A_j^+\},\ \{A_i^{\xi}, A_j^{\xi}\}, \quad i \neq j = 1, \ldots, n. \tag{45}$$

Observe that the Cartan-Weyl root vectors are expressed in terms of the pB operators exactly as in the nondeformed case.

In particular the operators

$$L_i^{\pm 1},\ \{A_i^-, A_j^+\},\ i \neq j = 1, \ldots, n \tag{46}$$

give a realisation of the quantum $gl(n)$ algebra in terms of deformed pB operators, which is exactly the same as in the nondeformed case.

Define a new set of operators

$$a_i^{\pm},\ l_i = \hat{q}^{N_i}\ i = 1, \ldots, n \tag{47}$$

where $q = \hat{q}^2$, which satisfy the relations

$$\begin{aligned} &a_i^- a_i^+ - \hat{q}^{\pm 2} a_i^+ a_i^- = \frac{2}{\hat{q} + \hat{q}^{-1}} l_i^{\mp 2},\ i = 1, \ldots, n, \\ &a_i^{\xi} a_j^{\eta} = \hat{q}^{2\xi\eta} a_j^{\eta} a_i^{\xi}, \quad i < j, \quad \xi,\ \eta = \pm,\ i = 1, \ldots, n. \end{aligned} \tag{48}$$

For fixed i the operators

$$a_i^{\pm},\ l_i = \hat{q}^{N_i} \tag{49}$$

are the same (up to multiple) as the deformed Bose CAOs [6-9]. The different modes however do not commute, they q-commute.

It is easy to check that the operators $a_i^{\pm},\ l_i$ satisfy the defining relations (44) of the deformed algebra $U_q[osp(1/2n)]$. Therefore we have

Proposition 6. The map

$$A_i^{\pm} \to a_i^{\pm} \quad L_i \to \hat{q}^{-1} l_i^{-2} \tag{50}$$

defines a representation, a kind of Fock representation of $U_q[osp(1/2n)]$.

In particular the operators

$$l_i,\ \{a_i^-, a_j^+\},\ i \neq j = 1, \ldots, n \tag{51}$$

give a realisation, a Schwinger realisation of the quantum $gl(n)$ algebra in terms of new kind of deformed Bose operators. Elsewhere we will study the Fock representations of the creation and the annihilation operators (48) and the related oscillator representations of $U_q[osp(1/2n)]$ and $U_q[gl(n)]$.

B. Second realization. Quantization in terms of Biedenharn-Macfarlane generators

It may be more convenient to quantize $U_q[osp(1/2n)]$ in terms of new generators, namely $(i = 1, \ldots, n)$

$$\begin{aligned}
B_i^- &= \hat{q}^{n-i}\sqrt{\frac{\hat{q}+\hat{q}^{-1}}{2\hat{q}}}A_i^- L_i^{-\frac{1}{2}} L_{i+1}^{-1} L_{i+2}^{-1} \ldots L_n^{-1},\\
B_i^+ &= \hat{q}^{i-n}\sqrt{\frac{\hat{q}+\hat{q}^{-1}}{2\hat{q}}}A_i^+ L_i^{\frac{1}{2}} L_{i+1} L_{i+2} \ldots L_n,\\
K_i &= \hat{q}^{-\frac{1}{2}} L_i^{-\frac{1}{2}}.
\end{aligned} \tag{52}$$

Proposition 7. $U_q[osp(1/2n)]$ is the (free unital) associative algebra with generators $B_i^{\pm}$, K_i, $i = 1, \ldots, n$ and the relations

$$\begin{aligned}
&K_i K_i^{-1} = K_i^{-1} K_i = 1, \ \ K_i K_j = K_j K_i,\\
&K_i B_j^{\pm} = \hat{q}^{\pm\delta_{ij}} B_j^{\pm} K_i, \quad i,j = 1, \ldots, n,\\
&\{B_i^-, B_i^+\} = \tfrac{\hat{q}K_i^2 - q^{-1}K_i^{-2}}{\hat{q}-\hat{q}^{-1}}, \quad i = 1, \ldots, n,\\
&[\{B_{i\pm1}^{-\eta}, B_i^{\eta}\}_{\hat{q}^{\mp 2}}, B_j^{-\eta}]_{\hat{q}^{\pm 2\delta_{ij}}} = -\eta(1+\hat{q}^{\pm 2\eta})\delta_{ij} K_j^{\pm 2\eta} B_{i\pm1}^{-\eta}, \quad \eta = \pm,\\
&[\{B_{n-1}^{\xi}, B_n^{\xi}\}_{\hat{q}^2}, B_n^{\xi}]] = 0, \quad \xi = \pm.
\end{aligned} \tag{53}$$

The reason to introduce the operators (52) stems from the next proposition.

Proposition 8. Let $b_i^{\pm}$, k_i be the deformed CAOs with $k_i = \hat{q}^{N_i}$ [6-9]:

$$\begin{aligned}
&b_i^- b_i^+ - \hat{q}^{\pm 2} b_i^+ b_i^- = k_i^{\mp 2},\\
&k_i b_i^{\pm} = \hat{q}^{\pm} b_i^{\pm} k_i
\end{aligned} \tag{54}$$

Assume that the different modes of such operators commute. Then the map

$$B_i^{\pm} \to b_i^{\pm}, \quad K_i \to k_i \tag{55}$$

defines a representation of $U_q[osp(1/2n)]$.

We call the generators (52) Biedenharn-Macfarlane generators of $U_q[osp(1/2n)]$. According to Proposition 7 the Biedenharn-Macfarlane generators give an alternative definition of the deformed orthosymplectic superalgebra $U_q[osp(1/2n)]$. The main algebraic feature of these generators stems from Proposition 8: in the Fock representation they coincide with the deformed creation and annihilation operator (54).

It is not easy to write down all triple relations the Biedenharn-Macfarlane operators satisfy. Here are some of them:

$$\begin{aligned}
&[\{B_i^-, B_j^+\}_{\hat{q}^2}, B_k^+] = 0, \quad i < j \leq k \text{ or } k < i < j, \\
&[\{B_i^-, B_j^+\}_{\hat{q}^2}, B_k^+]_{\hat{q}^4} = (1 - \hat{q}^4)\{B_i^-, B_k^+\}_{\hat{q}^2} B_j^+, \quad i < k < j, \\
&[\{B_i^-, B_j^+\}_{\hat{q}^2}, B_i^+]_{\hat{q}^2} = (1 + \hat{q}^2) B_j^+ K_i^{-2}, \ i < j, \\
&[\{B_i^-, B_j^+\}_{\hat{q}^2}, B_k^-] = 0, \quad k \leq i < j \text{ or } i < j < k, \\
&[\{B_i^-, B_j^+\}_{\hat{q}^2}, B_k^-]_{\hat{q}^{-4}} = (1 - \hat{q}^{-4})\{B_k^-, B_j^+\}_{\hat{q}^2} B_i^+, \quad i < k < j, \\
&[\{B_i^-, B_j^+\}_{\hat{q}^2}, B_j^-]_{\hat{q}^{-2}} = -(1 + \hat{q}^{-2}) B_i^- K_j^{-2}, \ i < j,
\end{aligned} \tag{56}$$

and the conjugate to the above relations:

$$\begin{aligned}
&[\{B_i^-, B_j^+\}_{\hat{q}^{-2}}, B_k^-] = 0, \quad j < i \leq k \quad \text{or} \quad k < j < i, \\
&[\{B_i^-, B_j^+\}_{\hat{q}^{-2}}, B_k^-]_{\hat{q}^4} = (1 - \hat{q}^4)\{B_k^-, B_j^+\}_{\hat{q}^{-2}} B_i^-, \quad j < k < i, \\
&[\{B_i^-, B_j^+\}_{\hat{q}^{-2}}, B_j^-]_{\hat{q}^2} = -(1 + \hat{q}^2) B_i^- K_j^2, \quad i > j, \\
&[\{B_i^-, B_j^+\}_{\hat{q}^{-2}}, B_k^+] = 0, \quad k \leq j < i \quad \text{or} \quad j < i < k, \\
&[\{B_i^-, B_j^+\}_{\hat{q}^{-2}}, B_k^+]_{\hat{q}^{-4}} = (1 - \hat{q}^{-4})\{B_i^-, B_k^+\}_{\hat{q}^{-2}} B_j^+, \quad j < k < i, \\
&[\{B_i^-, B_j^+\}_{\hat{q}^{-2}}, B_i^+]_{\hat{q}^{-2}} = (1 + \hat{q}^{-2}) B_j^+ K_i^2, \quad j < i.
\end{aligned} \tag{57}$$

Using the above relations we will show elsewhere that all operators k_i together with

$$\begin{aligned}
&\{b_i^-, b_j^+\}_{\hat{q}^2} \quad \text{for} \quad i < j, \\
&\{b_i^-, b_j^+\}_{\hat{q}^{-2}} \quad \text{for} \quad i > j
\end{aligned} \tag{58}$$

define a representation of Cartan-Weyl generators of $U_q[gl(n)]$ and we will study the corresponding oscillator representations.

4. Comments

We have not touched here the coalgebra and, more generally, the entire Hopf algebra structure of $U_q[so(2n+1)]$ and $U_q[osp(1/2n)]$. One can in principle derive the expression for the action of the comultiplication Δ on the creation and the annihilation operators using the known relations for the action of Δ on the Chevalley generator and the expressions for the preoscillator generators in terms of the Chevalley operators. The corresponding expressions are however extremely involved and we do not have any explicit formulae for $\Delta(B_i^{\pm})$ for instance. On the other hand one needs such relations in order to give a complete Hopf algebra description of $U_q[so(2n+1)]$ and $U_q[osp(1/2n)]$ in terms of preoscillator generators. Therefore we will conclude the present talk with a problem.

Problem. Find closed expressions for the action of the comultiplication and the antipode on the preoscillator generators.

5. Acknowledgments

It is a pleasure to thank Dr. J. Van der Jeugt, Dr. R. Jaganathan and Dr. N. I. Stoilova for the stimulating discussions. The author is thankful to Prof. Guido Vanden Berghe for the kind hospitality at the Department of Applied Mathematics and Computer Science, University of Ghent.

This work was supported by a grant awarded by the Belgian National Fund for Scientific Research and by the contract Φ-416 of the Committee of Science of Bulgaria.

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